AXIOM OF CHOICE'S CONSEQUENCES FOR METHODOLOGICAL MATERIAL IN MATHEMATICS TEACHING

Tomáš Zdráhal

Abstract

The article deals with one effect of the Axiom of Choice's bearing on understanding how to get better insight into traditional mathematical notions and concepts. It is shown that the investigation of a special function brings didactic and methodological mathematical material for the demonstration of the modern concept of teaching mathematics at universities.

Key words

Homomorphism, functional equation, Cauchy equation, Axiom of Choice, Hamel basis, discontinuous additive function.

The main idea of this article is to show, by means of a special non-trivial mathematical function, that it is not possible to raise mathematics as the science only on the basis of intuitive notions and concepts. To show, that the basic implement for an acquirement of knowledge in the mathematics is the language of mathematics and that this implement is so much preferred, that every knowledge obtained in different way, e.g. on the basis of an observation, an experiment, an intuition, a sense, etc. is considered to be the mathematical knowledge only when it is proved by means of the language. To show, that the acquirement of new knowledge only by consideration is characteristic of mathematics; this makes mathematics to be exact science in letter. Each other science is exact only to the extent, to which it uses just mathematics. Because of the scope of this journal there will not be any mathematics; only the didactic background of some mathematical consideration (for mathematics see e.g. [7]) in this article.

Let us start with the fact that just general linear function could be the very suitable function to demonstrate the facts above mentioned. In spite of our words saying there will be no mathematics in this paper, it is necessary to introduce at least few the most important mathematical concepts essential for our considerations understanding. What do we mean by general linear function then?

Definition

A function $f: \mathbf{R}^N \to \mathbf{R}$ *is called general linear iff it is of the form*

$$f(x) = f_a(x) + b,\ x \in \mathbf{R}^N,$$

where f_a *is a homomorphism of the linear space* $(\mathbf{R}^N, \mathbf{Q}, +, \cdot)$ *into the linear space* $(\mathbf{R}, \mathbf{Q}, +, \cdot)$ *and* $b = f(0)$.

Thus $f_a: \mathbf{R}^N \to \mathbf{R}$ is additive, i.e. it satisfies Cauchy's functional equation

$$f_a(x + y) = f_a(x) + f_a(y)$$

for all $x, y \in \mathbf{R}^N$.

Theorem

Let H be an arbitrary Hamel basis of the space $(\mathbf{R}^N, \mathbf{Q}, +, \cdot)$. *Then for every function* $g: H \to \mathbf{R}$ *there exists a unique additive function* $f: \mathbf{R}^N \to \mathbf{R}$ *such that* $f \mid H = g$.

This assertion gives the general construction of all additive functions $f: \mathbf{R}^N \to \mathbf{R}$. In fact, every additive f may be obtained as the unique additive extension of a certain function $g: H \to \mathbf{R}$ that is $g = f \mid H$.

It could be shown (see again [7] and [8]) additive (and general linear) functions bring didactic and methodological mathematical material for the demonstration of the modern concept of teaching mathematics, which is a good opportunity for the teacher to encourage and develop creative powers of students. Besides, mathematicians are as well attracted to functional equations by their apparent simplicity and harmonic nature, which may conceal the possibility of obtaining important mathematical results. Actually, the very theory of functional equations (and additive functions are solutions of the fundamental functional equation) raises a wide range of mathematical problems whose solutions have not only the formative effect, but also an informative one – in other fields of mathematics. The specificity of solving these problems lies in the fact that although we can use our already gained knowledge and skills in order to solve them heuristically; we cannot base our solutions on concrete geometric image however. We are therefore faced with the necessity to prove our hypotheses in an exact way, which is very valuable and desirable. Why cannot we rely on the geometric image here? Because our solutions come from the existence of discontinuous additive functions. There is necessary to put on an explanation here.

For many years the existence of discontinuous additive functions was an open problem. Mathematicians could neither prove that every additive function is continuous nor exhibit an example of discontinuous additive function. It was only thanks to the development of the set theory in the beginning of the 20th century that the existence of a discontinuous additive function could be proved. It was G. Hamel who first succeeded in proving this fact. The proof is based on the existence of a special set of elements from $\mathbf{R}^N$, today called a Hamel basis of $\mathbf{R}^N$. Its existence follows from the Axiom of Choice. It means however that no concrete example of such base is known, i.e. effective examples of discontinuous additive functions do not exist.

These results are not the very trivial. However, our goal is to investigate neither mathematical background of this phenomena nor the theory of Cauchy functional. We only want to outline against this background following (from the mathematics' didactic point of view very procreative) attribute: That newly acquired knowledge (i.e. active general linear function understanding) strengthens insight into traditional notions like the continuity of a function in a point, the continuity of a function on a set, the limit of a function, the monotony of a function, etc. From the pedagogy-psychological point of view there is the most important investigation and the consecutive solution of a reality here, namely that the question of existence of a discontinuous additive function cannot be resolved without the use of the Axiom of Choice. The mathematics teacher should proceed very carefully and reasonably here. Among others he has to convince his students that every problem in mathematics is at least desirable to solve even at the cost of additional premises. An answer like "It depends on an axiomatic system." is unacceptable. Obviously we cannot simply say that if we insert the Axiom of Choice to this system too, the discontinuous additive function will exist. Student's response could be: "I need no Axiom of Choice and therefore the discontinuous additive function does not exist for me." Fortunately (in this case) our hypothetical student virtually helped us with these contents. Namely if we do not accept the Axiom of Choice, we cannot assert that the additive discontinuous function exists, but also it does not exist! Again we appeal to a teacher's effort of the well-considered approach to similar situations. Because further his students will proceed in the "unconstructive mathematics". They cannot issue from any geometrical notions and that is why they will not be sure of their intuitive judgments. The training of proofs' techniques on theorems which are intuitively obvious, is boring and from students' point of view useless. These theorems are quite different – here students are really motivated for this not in favour activity. Here they both prove "their proposition" and they are not sure of its acceptance at all. Students can indeed notify here that it is not possible to construct mathematics on the basis of intuitive notions. Here students can appreciate at all, that a basic implement for an acquirement of knowledge in the mathematics is the language of mathematics and that this implement is so much preferred, that any knowledge obtained in another way is considered the mathematical knowledge, only when it is proved by means of the mathematical language. Obviously the contribution of this matter is great to all intents and purposes – owing to the Axiom of Choice, which is used in it. Now it already depends on a teacher which next consequences of this one of the most important axioms he will go into with students. Here we only refer to the fact that without the Axiom of Choice we would not prove the proposition:

The function f: $\mathbf{R} \to \mathbf{R}$ *is continuous in the point* x_0 *if and only if the following condition is satisfied for each sequence of real numbers* x_n:

$$\lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0).$$

There is no need to emphasize here how valuable from the didactic point of view is to find the moment in the proof of this known proposition, when the Axiom of Choice is used. Let us note here yet, that even the proof is mentioned in each textbook of mathematical analysis, the reality, that the Axiom of Choice is used in this proof, is not mentioned explicitly almost anywhere. To totalize, students thanks to our approach learn why we investigate e.g. additive functions. Not because they would be attractive for us themselves. But to find the weakest conditions, whose completion will guarantee, that we will be able to avoid them...

Now let us mention two applications of additive function yet, which certainly will extend a mathematical reach of our students.

First is the question of an axiomatic approach to the area of a rectangle Let us indicate $F(x, y)$ the area of the rectangle with sides x and y.

Let us require the area should be defined for any sides, should be always positive and equalled to 1 for unitary sides and finally equalled to the sum of areas of two rectangles, whose the area is a disjoint union.

In this problem in fact we look for the function F: $\mathbf{R}^2_+ \to \mathbf{R}$, which satisfies the system of two functional equations

$$F(x_1 + x_2, y) = F(x_1, y) + F(x_2, y)$$
$$F(x, y_1 + y_2,) = F(x, y_1) + F(x, y_2)$$

where $x, x_1, x_2, y, y_1, y_2 \in \mathbf{R}_+$. Its solution is in form $F(x, y) = cxy,$ from that we already get the known formula (the rectangle with sides 1 is to have the area 1)

$$F(x, y) = xy.$$

Second application. The very interesting example of using Cauchy functional equation was given by solving one of twenty-three Hilbert's problems. In the concrete, the 3rd Hilbert's problem, put forth by him at the Paris conference of the International Congress of Mathematicians in 1900:

Are two arbitrary polyhedra with the same volume equidecomposable?

The result for polygons is affirmative and was known for long time as the Bolyai–Gerwien theorem.

The answer on the 3rd Hilbert's problem was as follows:

The analogous theorem for polyhedra does not hold, since e.g. a regular tetrahedron and a cube of the same volume are not equidecomposable.

The problem was solved in such way, that is was proved by means of certain invariants, that a tetrahedron and a cube are not equivalent by dissection under any circumstances. In the proof the additive function f: $\mathbf{R} \to \mathbf{R}$ satisfying the conditions $f(1) = 0$ and $f(\alpha) = 1$, where α is an arbitrary fixed irrational number was used.

Immensely valuable thing appears from a didactical point of view here: The reality, that the invariants of the tetrahedron and the cube differs one from another, can be proved without former mentioned discontinuous additive function too – consequently without the Axiom of Choice. Thus the facts are demonstrated here, that although the constructive mathematics is very attractive from the aesthetical point of view (it namely recognizes only objects, which can be constructed explicitly), this one often complicates proofs of even simple propositions which are clear in classical (it means unconstructive) mathematics.

References

[1] J. Aczél: *Lectures on functional equations and their applications*, Academic press, New York – San Francisko – London, 1966

[2] M. Kuczma: An introduction to the theory of functional equations and inequalities, Cauchy's equation and Jensen's inequality, PWN, Warszawa – Kraków – Katowice, 1985

[3] F. Neuman: Fukcionální rovnice, Matematický seminář SNTL, Praha, 1986

[4] J. Smítal: O funkciách a funkcionálnych rovniciach, Knižnica Epsilon, Alfa, Bratislava, 1984

[5] T. Zdráhal: Generalizations of some theorems of Steinhaus, Aequationes Math 56/3 1998, 298

[6] T. Zdráhal: On some consequences of universal sets,Matematyka 6, Prace naukowe WSP Czestochowa 1999, 153–157

[7] T. Zdráhal: Zobecněné elementární funkce, Acta Universitatis Purkynianae 125, Studia Mathematica, Ustí nad Labem, 2006

[8] T. Zdráhal: Některé vlastnosti aditivní funkce, Matematika v škole dnes a zajtra, KU Ružomberok 2007, 333–338

Tomáš Zdráhal,
Departments of Mathematics of Faculties of Education,
Palacky University in Olomouc and J. E. Purkyne University in Usti n. L.,
Czech Republic
tomas.zdrahal@upol.cz

Vliv axiomu výběru na metodické materiály ve výuce matematiky

Tomáš Zdráhal

Abstrakt

Tento článek se zabývá jedním důsledkem axiomu výběru, který umožňuje pochopit, jak mužeme lépe porozumět tradičním matematickým pojmům. Je zde ukázáno, že vyšetřování jedné speciální funkce přináší didaktický a metodický materiál pro demonstraci moderní koncepce výuky matematiky na vysokých školách.

Klíčová slova

Homomorfismus, funkcionální rovnice, Cauchyova rovnice, axiom výběru, Hamelova báze, nespojitá aditivní funkce.